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HILBERT SPACES AND FOURIER SERIES

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Hilbert Spaces and Fourier Series

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Terri Joan Harris

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Abstract

I give an overview of the basic theory of Hilbert spaces necessary to understand the convergence of the Fourier series for square integrable functions. I state the necessary theorems and definitions to understand the formulations of the problem in a Hilbert space framework, and then I give some applications of the theory along the way.

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Chapter 1

Introduction

The main purpose of this thesis is to undertand the structure of a Hilbert space and to study and understand the proof of the convergence of Fourier series for square summable functions. The theory of Hilbert spaces provides a general context for many problems in pure and applied mathematics. While many are familiar with the finite-dimensional Euclidean spaces, functional analysis is concerned with infinite-dimensional spaces. Complete normed vector spaces over the real or complex numbers, called Banach spaces, are the foundation of this field of study. An important example is the Hilbert space where the norm arises from an inner product. L^2 is a Hilbert space and plays an important role in applications, especially in Fourier Analysis. The Fourier series arose from the study of infinite sums of sines and cosines and their solutions to physical problems. Joseph Fourier is credited with using the infinite sums of sines and cosines to model the behavior of heat flow. The Fourier series and its applications are important to the study of harmonic analysis and to quantum physics. Hilbert spaces seem accessible because many geometric intuitions from concepts, problems and their solutions, in Euclidean spaces do generalize to, and provide good pictures of, corresponding situations in Hilbert spaces. On the other hand, their applicability to many mathematical areas translates the geometric intuitions to new contexts and allows us to understand more sophisticated problems that originally seemed untractable, and unrelated to other mathematical problems.

We will study the structure of a Hilbert space in order to see that the Fourier series of square summable functions converges in the mean to its corresponding function. We are mostly interested in the convergence result as a consequence of the Hilbert space structure, and not in Fourier series per se. We will examine pointwise and uniform convergence, and see why they do not fit the convergence of Fourier series. Then we will examine how mean convergence is the desired type of convergence to use for the Fourier series of a square summable function, and how that motivates the introduction of the concept of Hilbert space. We will study several problems which illustrate the crucial steps of the convergence result for Fourier series. The first two problems will show how to compute the Fourier series of an even and an odd fucntion and examine why each series converges in mean to its corresponding function, and why pointwise and uniform convergence fail in these two cases. We will then examine the relationship between the three types of convergence, pointwise, uniform, and mean convergence. Another problem applies the Gram-Schmidt orthonormalization process to $1, x, x^2, ...$ to obtain formulas for the first four Legendre polynomials. The final problem will apply Parseval's identity to obtain the Euler's identity for the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and for the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$. Several theorems that assist in the development of Hilbert spaces and Fourier series will be included and discussed. We will conctruct proofs for the uniqueness part of the Projection Theorem, and for the Lemma used to prove said theorem. A proof of Parseval's Theorem is also included.

The following is a brief outline of the thesis. In Chapter 2 we will begin with the basic definitions and terminology that will assist the reader in understanding the material presented. Chapter 3 provides several examples of Hilbert spaces. Chapter 4 investigates the idea of convergences and discusses pointwise, uniform, and mean convergence. We demonstrate how to compute a Fourier series of a given function and discuss its type of convergence. The final example in Chapter 4 illustrates the Gram-Schmidt process which allows us to produce an orthonormal system. In Chapter 5, we discuss the importance of the Projection Theorem and its Lemma. Next, we state Bessel's Inequality and prove Parseval's Theorem. We conclude with Chapter 7 where we show within the Hilbert space, L^2, the Fourier series of a square summable function converges in the mean to its corresponding function.

Chapter 2

Basic Definitions and Terminology

The following definitions will provide the necessary background information and assist with understanding the terminology used in this paper. Although definitions are fairly standard, the bulk of the following definitions can be found in [Sax01].

A metric is a way of measuring distance between two points. In Euclidean space, $\mathbb{R}^n$, the metric is $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2}$ for $x = (x_1, x_2, ...x_n)$ and $y = (y_1, y_2, ...y_n)$ in $\mathbb{R}^n$.

More formally, a metric space is defined as follows.

Definition 2.1. *A metric space $(\mathcal{M}, d)$ is a set $\mathcal{M}$ together with a function $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ called a metric satisfying four conditions:*

1. $d(x, y) \geq 0$ *for all* $x, y \in \mathcal{M}$.
2. $d(x, y) = 0$ *if and only if* $x = y$.
3. $d(x, y) = d(y, x)$ *for all* $x, y \in \mathcal{M}$.
4. $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in \mathcal{M}$.

Definition 2.2. *Let $(\mathcal{M}, d)$ be a metric space. The open r-ball centered at x is the set $B_r(x) = \{y \in \mathcal{M} : d(x, y) < r\}$ for any choice of $x \in \mathcal{M}$ and $r > 0$. A closed r-ball centered at x is the set $\overline{B}_r(x) = \{y \in \mathcal{M} : d(x, y) \leq r\}$.*

The metric with which we will work with is induced by a norm. The definition of a norm is as follows:

Definition 2.3. *A (complex) normed linear space* $(\mathcal{V}, \|\cdot\|)$ *is a (complex) linear space* $\mathcal{V}$ *together with a function* $\|\cdot\| : \mathcal{V} \to \mathbb{C}$ *called a norm satisfying the following conditions:*

1. $\|v\| \geq 0$ *for all* $v \in \mathcal{V}$.
2. $\|v\| = 0$ *if and only if* $v = 0$.
3. $\|\lambda v\| = |\lambda| \, \|v\|$ *for all* $v \in \mathcal{V}$ *and* $\lambda \in \mathbb{C}$.
4. $\|v + w\| \leq \|v\| + \|w\|$ *for all* $v, w \in \mathcal{V}$.

When we have a normed vector space $(\mathcal{V}, \|\cdot\|)$, we can obtain a metric on $\mathcal{V}$ by setting $d(x, y) = \|x - y\|$ for $x, y \in \mathcal{V}$.

Definition 2.4. *A (complex) inner product space* $(\mathcal{V}, < \cdot, \cdot >)$ *is a (complex) linear space together with a function* $< \cdot, \cdot >: \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ *called an inner product satisfying the following conditions:*

1. $< v, v > \geq 0$ *for all* $v \in \mathcal{V}$.
2. $< v, v > = 0$ *if and only if* $v = 0$.
3. $< \lambda v, w > = \lambda < v, w >$ *for all* $v, w \in \mathcal{V}$ *and* $\lambda \in \mathbb{C}$.
4. $< v, \lambda w > = \overline{\lambda} < v, w >$ *for all* $v, w \in \mathcal{V}$ *and* $\lambda \in \mathbb{C}$.
5. $< v, w > = < \overline{w, v} >$ *for all* $v, w \in \mathcal{V}$.
6. $< v, w + u > = < v, w > + < v, u >$ *for all* $u, v, w \in \mathcal{V}$.
7. $< v + u, w > = < v, w > + < u, w >$ *for all* $u, v, w \in \mathcal{V}$.

The primary space we will work with is a Hilbert space. The definition of a Hilbert space is as follows:

Definition 2.5. *A Hilbert space is a vector space* H *with an inner product* $< f, g >$ *such that the norm defined by* $\|f\| = \sqrt{< f, f >}$ *turns* H *into a complete metric space. Complete means that the Cauchy sequences converge.*

The Trigonometric System is important in our discussion regarding the Fourier Series.

Definition 2.6. *The Trigonometric System on* $[-\pi, \pi]$ *is given by:*

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos(nx)}{\sqrt{\pi}}, \frac{\sin(mx)}{\sqrt{\pi}}, \; n, m = 1, 2, \ldots$$

The Fourier Series will be a major topic of discussion and is defined as follows:

Definition 2.7. *The Fourier Series for a function* f *on the interval* $[-\pi, \pi]$ *is*

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mt + b_m \sin mt)$$

Where

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) cos(mt) dt$$

and

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) sin(mt) dt.$$

Definition 2.8. *[Kat76]* $L^2(D)$*, is the set of complex valued functions* $f(t)$ *on the real number line with* $\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$*.* $L^2(D)$ *is known as the space of square integrable functions. Its inner product is defined as:* $< f, g >= \int_{-\infty}^{\infty} f(t)\overline{g(t)} dt$*.*

We are interested in the way the Fourier series of f converges to f. There are several possibilities to be considered. We define these types of convergence as follows:

Definition 2.9. *Let* f_n*,* $n = 1, 2, \ldots$ *and* f *be complex valued functions on a set* D*. The sequence* (f_n) *converges pointwise (on* D*) to the function* f *if for every* $x \in D$*, the sequence* $(f_n(x))_{n=1}^{\infty}$ *converges to* $f(x)$ *i.e.*

$$f_n(x) \longrightarrow f(x) \text{ as } n \to \infty.$$

Definition 2.10. *Let* f_n*,* $n = 1, 2, \ldots$ *and* f *be complex valued functions on a set* D*. The sequence* (f_n) *converges uniformly (on* D*) to the function* f *if for every* $\varepsilon > 0$*, the closed ball* $\overline{B}_{\varepsilon}(f)$ *absorbs the sequence* (f_n)*. i.e. For all* $\varepsilon > 0$*, there exists an* $N \in \mathbb{N}$ *such that for all* $n \geq N$ *and all* $x \in D$*, we have:*

$$|f_n(x) - f(x)| \leq \varepsilon.$$

Definition 2.11. *Convergence in the norm ("in mean" convergence)*

Let f_n, $n = 1, 2, \ldots$, and f be functions in $L^2(D)$. We say that the sequence (f_n) converges in norm if:

$$\lim_{n\to\infty} \|f_n - f\|_2 = 0.$$

In the case of the Fourier series of f, we have that the Fourier series of f converges to f in mean if:

$$\lim_{n\to\infty} \left[\int_{-\pi}^{\pi} \left[f(x) - \left(\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx) \right) \right]^2 dx \right] = 0.$$

In Chapter 7, we will show that this is the case, that is, for functions in $L^2(D)$, their Fourier series always converges in mean to their corresponding functions.

A useful geometric equality is the Parallelogram Identity.

Definition 2.12. *Let $\mathcal{V}$ be a normed linear space and x, y be elements of $\mathcal{V}$. We say that x and y satisfy the parallelogram identity if*

$$\|x + y\|^2 + \|x - y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2 .$$

This identity says that the sum of the squares of the lengths of the diagonals of a parrallelogram is twice the sum of the squares of the lengths of its sides which is a generalization of the Pythagorean theorem. The parrallelogram identity gives us a way to determine whether or not the norm is induced by an inner product.

Chapter 3

Some Examples of Hilbert Spaces

Recall that a Hilbert space is an infinite dimensional complete inner product space. The following are examples of Hilbert spaces.

Example 3.1. *The first example of a Hilbert space is* $\mathbb{R}^n$ *with the inner product,*
$\langle (x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \rangle = x_1y_1 + x_2y_2 + \cdots x_ny_n$.
This inner product induces the norm $\|(x_1, x_2, ...x_n)\|$ *and the metric*
$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}$.

Example 3.1 is a Euclidean space, that is a real Hilbert space. The examples that follow are all complex linear spaces.

Example 3.2. *The second example of a Hilbert space is* $\mathbb{C}^n$
where $v = (x_1 + iy_1, x_2 + iy_2, ..., x_n + iy_n)$ *with norm* $\left[\sum_{i=1}^{n}(x_i^2 + y_i^2)\right]^{\frac{1}{2}}$.

Example 3.3. *The third example of a Hilbert space is* l_2. l_2 *is a set of complex sequences* $(a_1, a_2, \ldots)$ *such that* $\sum |a_i|^2$ *converges and is called square summable. For* l_2 *we use the inner product of complex numbers* $< (a_1, a_2, \ldots), (b_1, b_2, \ldots) >= \sum_{j=1}^{\infty} a_j\overline{b_j}$.

Example 3.4. *Another example of a Hilbert space is* $L^2(\mathbb{R})$*, as defined in Definition* 2.8*, is the set of complex valued functions* $f(t)$ *on the real number line with* $\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$ *(that is,* f *is square integrable).* $L^2(\mathbb{R})$ *is known as the space of square integrable functions. Its inner product is defined as* $< f, g >= \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt$.

Example 3.5. *The final example example is* $L^2[a, b]$*, the set of complex valued functions on the interval* $[a, b]$ *such that* $\int_a^b |f(t)|^2 dt < \infty$ *where the inner product is defined as* $<$

$f, g >= \int_a^b f(t)\overline{g(t)}dt$. The Hilbert space $L^2[(-\pi, \pi)]$ is used for studying the trigonometric system and the Fourier Series.

Chapter 4

Types of Convergence

The general question with which we are concerned is: When does a Fourier series converge to its function? And if it does converge, what type of convergence? We will study the following types of convergence and discuss their relationships.

- Pointwise Convergence
- Uniform Convergence
- Mean Convergence

4.1 Uniform and Pointwise Convergence

Uniform convergence is the strongest type of convergence. Uniform convergence implies pointwise convergence. We will begin by looking at that proof.

Proposition 4.1. *Let D be a subset of $\mathbb{R}$, and $f_n, n = 1, 2, \ldots$ and f be complex valued functions on D. If $f_n \to f$ uniformly, then $f_n \to f$ pointwise.*

Proof. Let $D \subset \mathbb{R}$, $f_n : D \to \mathbb{R}(n \in \mathbb{N})$. Recall (Definition 2.8) that the function sequence $(f_n)_{n\in\mathbb{N}}$ is called uniform convergent to a limit function $f : D \to \mathbb{R}$ if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ for every $x \in D$ for every $n \in \mathbb{N}$ such that $n > \mathbb{N}$ implies that $|f_n(x) - f(x)| < \epsilon$. Therefore, N depends only on ϵ and not on x. Since N depends only on ϵ and not on x, then N can depend on both ϵ and x and therefore must be pointwise convergent as well.

□

However, pointwise convergence does not imply uniform convergence. Here is an example that illustrates this fact.

Example 4.1. *Let* $D = [0,1]$ *and* $f_n(x) = x^n$ *with*

$$f(x) = \begin{cases} 0 & \textit{if } 0 \leq x < 1 \\ 1 & \textit{if } x = 1 \end{cases}$$

Lets look at a few terms of the sequence. $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = x^3$, and $f_4(x) = x^4$.... It is easy to see that $f_n(x) \to 0 = f(x)$, for all $0 \leq x < 1$. Hence, we have pointwise convergence. However, if we let $\epsilon = \frac{1}{4}$, $\overline{B}_{\frac{1}{4}}(f)$ has no f_n inside because $f_n(x) = x^n$ is outside $\overline{B}_{\frac{1}{4}}(f)$, for $\frac{1}{\sqrt[n]{4}} < x < 1$. Therefore, the function sequence does not converge uniformly.

4.2 Mean Convergence

Uniform convergence implies convergence in mean.

Proposition 4.2. *Let* $f_n, f \in L^2(D)$. *If* $f_n \to f$ *uniformly, then* $f_n \to f$ *in mean.*

Proof. Since $f_n(x)$ is uniformly convergent to $f(x)$, by definition 2.8 we know that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in D$, we have

$$|f_n(x) - f(x)| \leq \varepsilon$$

Thus

$$|f_n(x) - f(x)|^2 < \epsilon^2 \text{ and } \int_a^b |f_n(x) - f(x)|^2 < \epsilon^2(b-a)$$

Hence $\lim\limits_{n\to\infty} \int_a^b |f_n(x) - f(x)|^2 \, dx = 0$ and therefore the sequence
$f_n(x)$ converges to $f(x)$ in mean. ☐

The following is an example that shows pointwise convergence does not imply convergence in mean.

Example 4.2.

$$f_n(x) = \begin{cases} \frac{1}{x} & \textit{if } \frac{1}{n} \leq x \\ 0 & \textit{if } 0 \leq x < \frac{1}{n} \end{cases}$$

$f_n(x)$ converges to

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x \\ 0 & \text{if } x = 0 \end{cases}$$

pointwise. However, $f_n(x)$ does not converge to $f(x)$ in mean since $\int_0^1 \frac{1}{x^2} dx = \infty$.

Now we give an example to show that convergence in mean does not imply pointwise convergence (and hence uniform convergence).

Example 4.3. *Let* $D = [0, 1]$ *with* $f_n(x) = x^n$ *and*

$$f(x) = \begin{cases} 0 & \textit{if } 0 \leq x < 1 \\ \frac{1}{2} & \textit{if } x = 1 \end{cases}$$

$f_n(x)$ converges to $f(x)$ in mean but $f_n(x)$ converges to $g(x)$ pointwise where

$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Thus, $f_n(x)$ does not converge to $f(x)$ pointwise even though $f_n(x)$ converges to $f(x)$ in mean.

4.3 Convergence of the Fourier Series

We will look now to the specific case of the Fourier series. First, a few definitions.

Definition 4.1. *[PZ97] The space* E *is the space of piecewise continuous functions on the interval* $[-\pi, \pi]$.

Definition 4.2. *[PZ97] The space* E' *is the space of of all functions* $f(x) \in E$ *such that the right-hand derivative,* $D_+ f(x)$*, exists for all* $-\pi \leq x < \pi$ *and the left-hand derivative,* $D_- f(x)$*, exists for all* $-\pi \leq x < \pi$.

Notice that every continuous function is a piecewise continuous function, that is, belonging to E. Every function which is differentiable on $[-\pi, \pi]$ belongs to E'. All functions in E belong to $L^2([-\pi, \pi])$.

In the case of the Fourier series we have the following theorem:

Theorem 4.3. *[PZ97] If $f \in E'$ and*

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k cos(kx) + b_k sin(kx))$$

then the series converges pointwise to

$$\frac{f(x_+) + f(x_-)}{2}.$$

That is

$$S_N(x) \to \frac{f(x_+) + f(x_-)}{2} \ \text{as } N \to \infty.$$

In particular, we have

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k cos(kx) + b_k sin(kx))$$

at every point $x \in [-\pi, \pi]$ where f(x) is continuous.

Proof. To prove this theorem, we will need the aid of several lemmas. In the proofs of lemmas $4.4, 4.5, 4.6$ and the proof of Theorem 4.3 we follow the ideas in [PZ97].

We will start with the following lemma.

Lemma 4.4.

$$a_k cos(kx) + b_k sin(kx) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)cos(kt)dt.$$

Proof. Using the definition of the constants, a_k, b_k, we obtain

$$a_k coskx + b_k sinkx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)[cos(kt)cos(kx) + sin(kt)sin(kx)]dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)cosk(t-x)dt$$

$$= \frac{1}{\pi}\int_{-(\pi-x)}^{\pi-x} f(s+x)cos(ks)ds$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(s+x)cos(ks)ds$$

Since

$$\int_{\alpha}^{\alpha+2\pi} F(s)ds = \int_{-\pi}^{\pi} F(s)ds$$

for all α for $F(s)$ with period 2π. This result with $F(s) = f(s+x)cos(ks)$ and $\alpha = -(\pi-x)$ gives us

$$\frac{1}{\pi}\int_{-(\pi-x)}^{\pi-x} f(s+x)cos(ks)ds = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s+x)cos(ks)ds.$$

Hence

$$a_k cos(kx) + b_k sin(kx) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)cos(kt)dt.$$

□

Lemma 4.4 and the fact that $\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)dt$, give

$$S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\left(\frac{1}{2} + \sum_{k=1}^{N} cos(kt)\right) dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)D_N(t)dt$$

where the Dirichlet's kernel is

$$D_N(t) = \left(\frac{1}{2} + \sum_{k=1}^{N} cos(kt)\right)$$

With the result of Lemma 4.4 and the notational changes, in order to prove Theorem 4.3, we show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)D_N(t)dt \to \frac{f(x_+)+f(x_-)}{2} \ as\ N \to \infty.$$

We achieve this by proving

$$\frac{1}{\pi}\int_{0}^{\pi} f(x+t)D_N(t)dt \to \frac{f(x_+)}{2}$$

and

$$\frac{1}{\pi}\int_{-\pi}^{0} f(x+t)D_N(t)dt \to \frac{f(x_-)}{2}$$

as $N \to \infty$.

Next we prove the formula:

Lemma 4.5.

$$D_N(t) = \frac{sin(N+\frac{1}{2})t}{2sin(\frac{t}{2})} \ for\ t \neq 0.$$

Proof. Notice that

$$D_N(t) = \frac{1}{2} + \sum_{k=1}^{N} cos(kt)$$

$$= \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{N}[e^{ikt} + e^{-ikt}]$$

$$\frac{1}{2}\left(1 + \sum_{k=1}^{N} e^{ikt} + \int_{k=1}^{N} e^{-ikt}\right).$$

Since $\sum_{k=1}^{N} q^k = \frac{1-q^{N+1}}{1-q}$ whenever $q \neq 1$, for $t \neq 0$, we put $q = e^{it}$ to obtain

$$\sum_{k=1}^{N} e^{ikt} = \frac{e^{it} - e^{(N+1)it}}{1 - e^{it}}$$

$$= \frac{e^{it} - e^{(N+1)it}}{e^{\frac{it}{2}}(e^{\frac{-it}{2}} - e^{\frac{it}{2}})}$$

$$= -\frac{e^{\frac{it}{2}} - e^{(N+\frac{1}{2})it}}{2isin(\frac{t}{2})}$$

Similiarly, with $q = e^{-it}$, we obtain

$$\sum_{k=1}^{N} e^{-ikt} = \frac{e^{\frac{-it}{2}} - e^{-(N+\frac{1}{2})it}}{2isin(\frac{t}{2})}.$$

Hence,

$$\sum_{k=1}^{N} e^{ikt} + \sum_{k=1}^{N} e^{-ikt} = \frac{1}{2isin(\frac{t}{2})}[e^{\frac{-it}{2}} - e^{\frac{it}{2}} + e^{(N+\frac{1}{2})it} + e^{-(N+\frac{1}{2})it}]$$

$$= \frac{1}{2isin(\frac{t}{2})}[-2isin(\frac{it}{2}) + 2isin(N + \frac{1}{2})it]$$

$$= -1 + \frac{sin(N+\frac{1}{2})it}{sin(\frac{t}{2})}$$

Thus,

$$D_N(t) = \frac{sin(N+\frac{1}{2})it}{sin(\frac{t}{2})}$$

for $t \neq 0$. We can calculate that $D_N(0) = N + \frac{1}{2}$, and since $D_N(t) \to D_N(0)$ as $t \to 0$, we have

$$D_N(t) = \begin{cases} \frac{sin(N+\frac{1}{2})it}{sin(\frac{t}{2})}, & t \neq 0 \\ N + \frac{1}{2}, & t = 0. \end{cases}$$

$\square$

Now we define two functions $g(t)$ and $h(t)$ by

$$g(t) = \frac{f(x+t) - f(x_+)}{2sin(\frac{t}{2})} \text{ and } h(t) = \frac{f(x+t) - f(x_-)}{2sin(\frac{t}{2})}.$$

The function $g(t)$ is piecewise continuous on $[0, \pi]$ and $h(t)$ is piecewise continuous on $[-\pi, 0]$. Thus,

$$\lim_{t \to 0_+} g(t) = \lim_{t \to 0_+} \frac{f(x+t) - f(x_+)}{t} \cdot \frac{t}{2sin(\frac{t}{2})} = D_+ f(x)$$

as well as

$$\lim_{t \to 0_-} h(t) = \lim_{t \to 0_-} \frac{f(x+t) - f(x_-)}{t} \cdot \frac{t}{2sin(\frac{t}{2})} = D_- f(x)$$

The limit exists because $f \in E'$. These results yield

$$\lim_{N \to \infty} \frac{1}{\pi} \int_0^\pi f(x+t) D_N(t) dt = \frac{f(x_+)}{2} \tag{4.1}$$

and

$$\lim_{N \to \infty} \frac{-\pi}{0} \int_0^\pi f(x+t) D_N(t) dt = \frac{f(x_-)}{2}. \tag{4.2}$$

It follows that

$$\lim_{N \to \infty} S_N(x) = \frac{1}{\pi} \int_{-\pi}^\pi f(x+t) D_N(t) dt = \frac{f(x_+) + f(x_-)}{2}$$

We prove the limits (4.1) and (4.2) in the following lemma.

Lemma 4.6.

$$\lim_{N \to \infty} \frac{1}{\pi} \int_0^\pi f(x+t) D_N(t) dt = \frac{f(x_+)}{2}$$

and

$$\lim_{N \to \infty} \frac{1}{\pi} \int_{-\pi}^0 f(x+t) D_N(t) dt = \frac{f(x_-)}{2}$$

Proof. First note that

$$\int_0^\pi D_N(t) dt = \frac{\pi}{2}, \text{ and } \int_{-\pi}^0 D_N(t) dt = \frac{\pi}{2}.$$

Then,

$$\begin{aligned} \frac{1}{\pi} \int_0^\pi f(x+t) D_N(t) dt - \frac{f(x_+)}{2} &= \frac{1}{\pi} \int_0^\pi f(x+t) D_N(t) dt - \frac{1}{\pi} \int_0^\pi f(x_+) D_N(t) dt \\ &= \frac{1}{\pi} \int_0^\pi (f(x+t) - f(x_+)) D_N(t) dt \end{aligned}$$

$$= \frac{1}{\pi}\int_0^{\pi} 2sin(\frac{t}{2})g(t)D_N(t)dt$$
$$= \frac{1}{\pi}\int_0^{\pi} g(t)sin(N+\frac{1}{2})tdt.$$

Next, we need to apply the Riemann-Lebesgue Lemma [Rud74], 5.14, which in our context simply says that the Fourier coefficients of f, a_n and b_n, satisfy that $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$. We claim that

$$\int_0^{\pi} g(t)sin(N+\frac{1}{2})tdt \to 0 \text{ as } N \to \infty$$

Notice that this result will yield

$$\frac{1}{\pi}\int_0^{\pi} f(x+t)D_N(t)dt - \frac{f(x_+)}{2} \to 0 \text{ as } N \to \infty,$$

that is

$$\lim_{N\to\infty} \frac{1}{\pi}\int_0^{\pi} f(x+t)D_N(t)dt = \frac{f(x_+)}{2}$$

Therefore establishing the first part of Lemma 4.6. In order to verify our claim, namely that

$$\int_0^{\pi} g(t)sin(N+\frac{1}{2})tdt \to 0 \text{ as } N \to \infty,$$

set

$$G(t) = \begin{cases} g(t) & : 0 < t \leq \pi \\ 0 & : -\pi \leq t \leq 0. \end{cases}$$

Since $g(t)$ is piecewise continuous on $[0, \pi]$, it follows that $G(t)$ is piecewise continuous on $[\pi, \pi]$. Hence $G(t) \in L^2([-\pi, \pi])$ and the Riemann-Lebesgue Lemma yields

$$\int_0^{\pi} g(t)sin(N+\frac{1}{2})tdt = \int_{-\pi}^{\pi} G(t)sin(N+\frac{1}{2})tdt \to 0 \text{ as } N \to \infty.$$

Which proves our claim and establishes the first equality in Lemma 4.6.
A similar calculation on $[-\pi, 0]$ using the function $h(t)$ defined before the statement of Lemma 4.6 established that

$$\lim_{N\to\infty} \frac{1}{\pi}\int_{-\pi}^{0} f(x+t)D_N(t)dt = \frac{f(x_-)}{2}.$$

and concludes the proof of lemma 4.6.

$\square$

As noted before Lemma 4.5, the result of Lemma 4.6 and Lemma 4.3 prove Theorem 4.3. Therefore, the Fourier series of f converges pointwise to $\frac{f(x_+)+f(x_-)}{2}$.

$\square$

Examples 4.4 and 4.5 show how to compute a classical Fourier series followed by a discussion about their convergence properties.

Example 4.4. *Given the function:*

$$f(x) = \begin{cases} 1 & : -\pi \leq x < 0 \\ 0 & : 0 \leq x < \pi \end{cases}$$

Show that its Fourier series is:

$$\frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{((-1)^n - 1)}{n} sin(nx)$$

Recall the Fourier series is given by:

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m cos(mt) + b_m sin(mt))$$

The Fourier coefficients of f for $k \geq 1$ are given by:

$$a_k = \frac{\langle f, cos(kt)\rangle}{\langle cos(kt), cos(kt)\rangle} \text{ and } b_k = \frac{\langle f, sin(kt)\rangle}{\langle sin(kt), sin(kt)\rangle}$$

$$where\, \langle cos(kt), cos(kt)\rangle = \pi \text{ and } \langle sin(kt), sin(kt)\rangle = \pi$$

To see why this is true, let $k = 1$ and therefore

$$\langle cost, cost\rangle = \int_{-\pi}^{\pi} cos^2 t dt$$

Since, $cos^2 t = \frac{1}{2}cos(2t) + \frac{1}{2}$, we have,

$$\int_{-\pi}^{\pi} \frac{1}{2}cos(2t) + \frac{1}{2} = \left[\frac{1}{4}sin(2t) + \frac{1}{2}t\right]_{-\pi}^{\pi} = \pi$$

Similarly, this is also true for $\langle sin(kt), sin(kt)\rangle$.

When $k = 0$,

$$\frac{a_0}{2} = \frac{\langle f, cos0 \rangle}{\langle cos0, cos0 \rangle} = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \frac{\pi}{2\pi} = \frac{1}{2}$$

Therefore the first term in the Fourier series is $\frac{a_0}{2} = \frac{1}{2}$.

When $k \geq 1$, then,

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) cos(kt) dt \text{ and } b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) sin(kt) dt$$

For $k \geq 1$, we have

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{0} f(t) cos(kt) dt + \frac{1}{\pi} \int_{0}^{\pi} f(t) cos(kt) dt \\ &= \frac{1}{\pi} \int_{-\pi}^{0} cos(kt) dt + \frac{1}{\pi} \int_{0}^{\pi} 0 dt \\ &= \left[\frac{1}{kt} sin(kt) \right]_{-\pi}^{0} + 0 = 0 \end{aligned}$$

Thus, $a_k = 0$ for all $k \geq 1$ since $sin(k\pi) = 0$

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi}^{0} f(t) sin(kt) dt + \frac{1}{\pi} \int_{0}^{\pi} f(t) sin(kt) \\ &= \frac{1}{-\pi} \int_{-\pi}^{0} sin(kt) dt + \frac{1}{\pi} \int_{0}^{\pi} 0 dt \\ &= \left[\frac{-1}{k\pi} cos(kt) \right]_{-\pi}^{0} \end{aligned}$$

$$= \frac{-1}{k\pi} \left[cos0 + cos(k\pi) \right] = \frac{-1}{k\pi} \left[1 \pm cos(k\pi) \right] \text{ depending on the value of k.}$$

$$cos\,(k\pi) = \begin{cases} -1 & : \text{k is even} \\ 1 & : \text{k is odd} \end{cases}$$

When k is odd, $b_k = 0$. When k is even, then b_k is the series:

$$\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n} sin(nx).$$

Hence, the Fourier Series for $f(x)$ is:

$$\frac{1}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n}sin(nx)$$

If we assume this series converges in mean to f, we should have:

$$\lim_{n\to\infty}\left[\int_{-\pi}^{\pi}\left[f(x)-\left(\frac{1}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n}sin(nx)\right)\right]^2dx\right]=0$$

We will first evaluate:

$$\left\|f(x)-\left(\frac{1}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n}sin(nx)\right)\right\|^2=$$

$$\left\langle f(x)-\left(\frac{1}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n}sin(nx)\right),f(x)-\left(\frac{1}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n}sin(nx)\right)\right\rangle$$

$$=\langle f,f\rangle-2\left\langle f,\frac{1}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n}sin(nx)\right\rangle$$

$$+\left\langle\frac{1}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n}sin(nx),\frac{1}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n}sin(nx)\right\rangle$$

$$=\langle f,f\rangle-\left\langle f,1+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n}sin(nx)\right\rangle$$

$$+\left\langle\frac{1}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n}sin(nx),\frac{1}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n}sin(nx)\right\rangle=$$

$$\langle f,f\rangle-\langle f,1\rangle-\left\langle f,\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n}sin(nx)\right\rangle+\int_{-\pi}^{\pi}\left|\frac{1}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n}sin(nx)\right|^2dx$$

$$=-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n}\int_{-\pi}^{0}sin(nx)dx$$

$$+\int_{-\pi}^{\pi}\left(\frac{1}{4}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{n}sin(nx)+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{(-1)^n-1}{n}\right)^2sin^2(nx)\right)dx$$

(Since $\langle f,f\rangle=\langle f,1\rangle=\pi$.)

$$=-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n}\left[\frac{-1}{n}cos(nx)\right]_{-\pi}^{0}+\frac{1}{4}(2\pi)+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{(-1)^n-1)}{n}\right)^2\int_{-\pi}^{\pi}sin^2(nx)$$

$(Since \int_{-\pi}^{\pi} sin(nx)dx = 0)$.

$$= \frac{-2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n}\left(\frac{-1}{n}+\frac{1}{n}(-1)^n\right)+\frac{\pi}{2}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{-1)^n-1}{n}\right)^2\cdot\pi$$

$$= \frac{-2}{\pi}\sum_{n=1}^{\infty}\left(\frac{2}{2n+1}\right)^2+\frac{\pi}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{2}{2n+1}\right)^2$$

Thus,

$$\frac{\pi}{2}-\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{2}{2n+1}\right)^2=0$$

And,

$$\frac{\pi}{2}=\frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}\right)^2$$

$$\frac{\pi^2}{8}=\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}\right)^2$$

Hence, the series converges in mean.

Now that we see that this series converges in mean, what about the pointwise and uniform convergence of this series? This series is neither pointwise or uniformly convergent. It is not pointwise convergent because f_n does not converge to f at $x = 0$ due to the discontinuity there. In fact, f_n converges to $\frac{1}{2}$ at $x = 0$. The series is not uniformly convergent due to the jump discontinuity at $x = 0$. Which means I can find $\overline{\mathbb{B}_\epsilon}(f)$ that does not absorb the whole series (i.e. $\epsilon = \frac{1}{4}$) which implies that f_n does not converge to f uniformly. Furthermore, since all the partial sums of the Fourier series of f are continuous functions, then this would imply that f would have to be continuous (which it is not) in order for the series to converge uniformly.

Example 4.5. *Let $f(x) = x^2$. Show that its Classical Fourier series is*

$$\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}cos(nx)$$

For $k = 0$, the first term of the series is:

$$\frac{a_0}{2}=\frac{\langle f, cos0\rangle}{\langle cos0, cos0\rangle}=\frac{\left\langle x^2, 1\right\rangle}{\langle 1, 1\rangle}=\frac{\frac{2\pi^3}{3}}{2\pi}=\frac{\pi^2}{3}$$

Thus, $\frac{a_0}{2} = \frac{\pi^2}{3}$ is the first term of the Fourier series.

For $k \geq 1$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 cos(kx)dx = \frac{1}{\pi}\left[\frac{(k^2x^2-2)sin(kx)+2kxcos(kx)}{k^3}\right]_{-\pi}^{\pi} =$$

$$\frac{1}{\pi}\left[\frac{(k^2\pi^2-2)sin(k\pi)+2k\pi cos(k\pi)}{k^3}\right]$$

$$-\frac{1}{\pi}\left[\frac{k^2(-\pi)^2-2)sin(-k\pi)+2k(-\pi)cos(-k\pi)}{k^3}\right]$$

$$= \frac{1}{\pi}\left\{\frac{2k\pi cos(k\pi)}{k^3} - \frac{-2k\pi cos(k\pi)}{k^3}\right\} = \frac{4cos(k\pi)}{k^2}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 sin(kx)dx = \frac{1}{\pi}\left[\frac{(2-k^2x^2)cos(kx)+2kxsin(kx)}{k^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(2-\pi^2k^2)cos(k\pi)+2k\pi sin(k\pi)}{k^3} - \frac{(2-\pi^2k^2)(cos(-k\pi))-2k\pi sin(-k\pi)}{k^3}\right] = 0$$

Thus, $b_k = 0$ for all k. Therefore, the classic Fourier series is

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}cos(nx)$$

If we assume this series converges in mean to f, we should have:

$$\lim_{n\to\infty}\left[\int_{-\pi}^{\pi}\left[f(x)-\left(\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}cos(nx)\right)\right]^2 dx\right] = 0$$

We will first evaluate:

$$\left\|f(x)-\left(\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}cos(nx)\right)\right\|^2$$

$$= \left\langle f(x)-\left(\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}cos(nx)\right), f(x)-\left(\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}cos(nx)\right)\right\rangle$$

$$= \langle f, f\rangle - 2\left\langle f, \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}cos(nx)\right\rangle$$

$$+\left\langle \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}cos(nx), \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}cos(nx)\right\rangle$$

$$= \left\langle x^2, x^2\right\rangle - \left\langle x^2, \frac{2\pi^2}{3}+8\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}cos(nx)\right\rangle$$

$$+\left\langle \frac{\pi^4}{9} + \frac{8\pi^2}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}cos(nx) + 16\sum_{n=1}^{\infty}\frac{1}{n^4}cos^2(nx)\right\rangle$$

$$= \int_{-\pi}^{\pi} x^4 dx - \int_{-\pi}^{\pi}\frac{2\pi^2}{3}x^2 dx - 8\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_{-\pi}^{\pi}x^2 cos(nx)dx$$

$$+ \int_{-\pi}^{\pi}\frac{\pi^4}{9}dx + \frac{8\pi^2}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_{-\pi}^{\pi}cos(nx)dx + 16\sum_{n=1}^{\infty}\frac{1}{n^4}\int_{-\pi}^{\pi}cos^2(nx)dx$$

$$= \frac{2\pi^5}{5} - \frac{4\pi^5}{9} - 32\pi\sum_{n=1}^{\infty}\frac{1}{n^4} + \frac{2\pi^5}{9} + 0 + 16\pi\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$= \frac{8\pi^5}{45} - 16\pi\sum_{n=1}^{\infty}\frac{1}{n^4}$$

Thus,

$$\frac{8\pi^5}{45} - 16\pi\sum_{n=1}^{\infty}\frac{1}{n^4} = 0$$

And,

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4}$$

Hence, the series converges in mean.

In addition, since $f(x) = x^2$ is continuous and real-valued and $\langle f, f_k\rangle = 0$ for each k, then f_n converges to f uniformly. Furthermore, by Proposition 4.1, $f_n(x)$ converges to $f(x)$ pointwise.

In order to insure that the Fourier series of f converges to f, it is important that we have an orthonormal basis. The Trigonometric System on $[-\pi, \pi]$ given by:

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos(nx)}{\sqrt{\pi}}, \frac{\sin(mx)}{\sqrt{\pi}},\ n, m = 1, 2, \ldots$$

is an orthonormal basis since:

$$\int_{-\pi}^{\pi}\left(\frac{1}{\sqrt{2\pi}}\right)^2 dx = 1$$

And,

$$\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}cos(nx)dx = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}sin(mx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}cos(nx)cos(mx)dx = 0$$

And,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}cos(nx)cos(mx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}sin(nx)sin(mx)dx$$

$$= \begin{cases} 1 & if\ n = m \\ 0 & if\ n \neq m \end{cases}$$

The Gram-Schmidt Orthonormalization process is an algorithm for producing an orthogonal or orthonormal basis for any nonzero subspace of $\mathbb{R}^n$. If $[u_1, u_2, ...]$is a basis for $\mathcal{V}$ (not necessarily orthogonal or normal), then we can obtain an orthonormal basis from it by taking

$$v_1 = \frac{u_1}{\|u_1\|}$$

and

$$v_2 = \left[u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1 \right]$$

and so on.

In general, if $\{v_1, v_2, ..., v_n\}$ is an orthonormal basis for the span $\{u_1, u_2, ..., u_n\}$, we take

$$v_{n+1} = u_{n+1} - \sum_{i=1}^{n} \frac{\langle u_{n+1}, v_i \rangle}{\|v_i\|^2} v_i$$

$\left(\sum_{i=1}^{n} \frac{\langle u_{n+1}, v_i \rangle}{\|v_i\|^2} v_i \text{ is the orthogonal projection of } u_{n+1} \text{ onto the span of } \{v_1, v_2, ..., v_n\} \right)$

This process is illustrated in the following example.

Example 4.6. *Apply the Gram-Schmidt process to the functions* $1, x, x^2, x^3, ...$*to obtain formulas for the first four Legendre polynomials. Then verify that they are indeed given by the formula*

$$\sqrt{\frac{2n+1}{2}} \cdot \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}(x^2 - 1)^n, n = 0, 1, 2, 3$$

To find the first Legendre polynomial $q_0(x)$, we begin with $u_0 = 1$ and $u_1 = x$. Since 1 and x are orthogonal, we need only to normalize the first polynomial.
Thus, since $\|1\|_2^2 = \int_{-1}^{1} 1dx = x|_{-1}^{1} = 2$, then $\|1\| = \sqrt{2}$ and $q_0(x) = 1 \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$.

To find the second Legendre polynomial $q_1(x)$ we apply the Gram-Schmidt process. $q_1'(x) = x - \frac{<x, \frac{1}{\sqrt{2}}>}{<\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}>} \cdot \frac{1}{\sqrt{2}} = x$ (since $< x, \frac{1}{\sqrt{2}} >= 0$). Next, we normalize and the result is $q_1(x) = x \cdot \frac{1}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}} x$ (since $\|x\|_2^2 = \int_{-1}^{1} x^2 dx = \frac{1}{3} x^3|_{-1}^{1} = \frac{2}{3}$).

Applying the Gram-Schmidt process again to find the third Legendre polynomial, $q_2(x)$, we compute $q_2'(x) = x^2 - \frac{<x^2, \frac{1}{\sqrt{2}}>}{<\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}>} \cdot \frac{1}{\sqrt{2}} - \frac{<x^2, \sqrt{\frac{3}{2}}x>}{<\sqrt{\frac{3}{2}}x, \sqrt{\frac{3}{2}}x>} \cdot \sqrt{\frac{3}{2}}x = x^2 - \frac{1}{3}$ which when scaled becomes $q_2'(x) = 3x^2 - 1$. Now we normalize and the result is $q_2(x) = (3x^2 - 1) \cdot \frac{1}{\|q_2'\|} = \sqrt{\frac{5}{8}}(3x^2 - 1) = \sqrt{\frac{5}{2}} \cdot \frac{1}{2}(3x^2 - 1)$

To find the fourth Legendre polynomial, $q_3(x)$, we compute $q_3'(x) = x^3 - \frac{<x^3, \frac{1}{\sqrt{2}}>}{<\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}>} \cdot \frac{1}{\sqrt{2}} - \frac{<x^3, \sqrt{\frac{3}{2}}x>}{<\sqrt{\frac{3}{2}}x, \sqrt{\frac{3}{2}}x>} \cdot \sqrt{\frac{3}{2}}x - \frac{<x^3, \sqrt{\frac{5}{8}}(3x^2-1)>}{<\sqrt{\frac{5}{8}}(3x^2-1), \sqrt{\frac{5}{8}}(3x^2-1)>} \cdot \sqrt{\frac{5}{8}}(3x^2 - 1)$ $= x^3 - \frac{3}{5}x$ which when scaled becomes $q_3'(x) = 5x^3 - 3x$. Lastly, we normalize and the result is $q_3(x) = (5x^3 - 3x) \cdot \frac{1}{\|q_3'(x)\|} = (5x^3 - 3x) \cdot \frac{1}{\sqrt{\frac{7}{8}}} = \sqrt{\frac{7}{2}} \cdot \frac{1}{2}(5x^3 - 3x)$

Now we will verify that they are indeed given by the formula:

$$\sqrt{\frac{2n+1}{2}} \cdot \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}(x^2 - 1)^n, n = 0, 1, 2, 3$$

$n = 0$:

$$\sqrt{\frac{1}{2}} \cdot \frac{1}{1 \cdot 0!} \cdot 1 = \frac{1}{\sqrt{2}}$$

$n = 1$:

$$\sqrt{\frac{2+1}{2}} \cdot \frac{1}{2 \cdot 1!} \cdot \frac{d}{dx}(x^2 - 1) = \sqrt{\frac{3}{2}} \cdot \frac{1}{2}(2x) = \sqrt{\frac{3}{2}} \cdot x$$

$n = 2$:

$$\sqrt{\frac{4+1}{2}} \cdot \frac{1}{2^2 \cdot 2!} \cdot \frac{d^2}{dx^2}(x^2 - 1)^2 = \sqrt{\frac{5}{2}} \cdot \frac{1}{8}(12x^2 - 4) = \sqrt{\frac{5}{2}} \cdot \frac{1}{2}(3x^2 - 1)$$

$n = 3$:

$$\sqrt{\frac{6+1}{2}} \cdot \frac{1}{2^3 \cdot 3!} \cdot \frac{d^3}{dx^3}(x^2 - 1)^3 = \sqrt{\frac{7}{2}} \cdot \frac{1}{48}(120x^3 - 72x) = \sqrt{\frac{7}{2}} \cdot \frac{1}{2}(5x^3 - 3x)$$

Chapter 5

The Projection Theorem

Hilbert spaces possess a very important geometric property that make them desirable to work with. If we have a closed subspace $\mathcal{M}$ of a given Hilbert space $\mathcal{H}$, then let $\mathcal{M}^{\perp}$ be the set of vectors in $\mathcal{H}$ that are orthogonal to $\mathcal{M}$. $\mathcal{M}^{\perp}$ is called the orthogonal complement of $\mathcal{M}$ and is also a Hilbert space. Thus $\mathcal{H} = \mathcal{M} + \mathcal{M}^{\perp} = \{x + y : x \in \mathcal{M}, y \in \mathcal{M}^{\perp}\}$. The following lemma and the projection theorem are very important in the further study of Hilbert spaces.

Lemma 5.1. *[RS80] and [Hal57] Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M}$ a closed subspace of $\mathcal{H}$, and suppose $x \in \mathcal{H}$. Then there exists in $\mathcal{M}$ a unique element z closest to x.*

Proof. Let $d = inf_{y \in \mathcal{M}} \|x - y\|$. Choose a sequence $\{y_n\}$, $y_n \in \mathcal{M}$, so that $\|x - y_n\| \to d$. We will show that $\{y_n\}$ is Cauchy. Pick two elements y_n and y_m of the sequence $\{y_n\}$. Let $u = y_n - x$ and $v = y_m - x$ then $u + v = -2x + y_n + y_m$ and $u - v = y_n - y_m$. Then

$$\|y_n - y_m\|^2 = \|(y_n - x) - (y_m - x)\|^2$$

$$= 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - \|-2x + y_n + y_m\|^2$$

by the parallelogram idenity, which becomes

$$= 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\left\|x - \frac{1}{2}(y_n + y_m)\right\|^2$$

$$\leq 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4d^2$$

(since $\|x - y_n\| \to d$, $\|x - y_m\| \to d$ and $\frac{1}{2}(y_n + y_m) \in \mathcal{M}$,
$\left\|x - \frac{1}{2}(y_m + y_n)\right\|^2 \geq d$)

As $n \to \infty$ and $m \to \infty$, $2d^2 + 2d^2 - 4d^2 = 0$. Therefore, $\|y_n - y_m\|$ converges to 0. Thus $\{y_n\}$ is Cauchy and since $\mathcal{M}$ is closed, $\{y_n\}$ converges to an element z of $\mathcal{M}$. Since $\|x - y_n\| \to d$ and $y_n \to z$, it follows easily that $\|x - z\| = d$

To show that z is unique, consider another projection $\widehat{z} \in \mathcal{M}$ such that $\|x - \widehat{z}\| = d$. Then we have

$$\|z - \widehat{z}\|^2 = \|(z - x) - (\widehat{z} + x)\|^2$$
$$= 2\|z - x\|^2 + 2\|\widehat{z} - x\| - \left\|x - \frac{1}{2}(z + \widehat{z})\right\|^2$$
$$= 2d^2 + 2d^2 - 4\left\|x - \frac{1}{2}(z + \widehat{z})\right\|^2$$
$$\leq 2d^2 + 2d^2 - 4d^2 = 0$$

Hence $\|z - \widehat{z}\|^2 = 0$. By the argument above, this is a Cauchy sequence and converges to an element z of $\mathcal{M}$, thus $z = \widehat{z}$. □

Theorem 5.2. *(The Projection Theorem) [RS80] Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M}$ a closed subspace. Then every $x \in \mathcal{M}$ can be uniquely written $x = z + w$ where $z \in \mathcal{M}$ and $w \in \mathcal{M}^{\perp}$*

Proof. Let x be in $\mathcal{H}$. Then by the lemma, there is a unique element $z \in \mathcal{M}$ closest to x. Define $w = x - z$, then we have $x = z + w$. Let $y \in \mathcal{M}$ and $t \in \mathbb{R}$. If $d = \|x - z\|$, then

$$d^2 \leq \|x - (z + ty)\|^2 = \|w - ty\|^2 = d^2 - 2tRe(w, y) + t^2\|y\|^2$$

Thus, $-2tRe(w, y) + t^2\|y\|^2 \geq 0$ for all t, and $Re(w, y) = 0$. Similarly, substituting ti instead of t produces $Im(w, y) = 0$. Hence, $w \in \mathcal{M}^{\perp}$.

To show uniqueness, we need to show that we have a unique z and w. Choose $z_1 \in \mathcal{M}$ and $w_1 \in \mathcal{M}^{\perp}$. We have $x = z + w = z_1 + w_1$. Thus, $x = z - z_1 = w_1 - w$. Since $z - z_1 \in \mathcal{M}$ and $w_1 - w \in \mathcal{M}^{\perp}$, the only element in both $\mathcal{M}$ and $\mathcal{M}^{\perp}$ is 0. Hence, $z - z_1 = 0$ and $w_1 - w = 0$, so $z = z_1$ and $w = w_1$.

□

The Projection Theorem contends that the closest function to f in the span of the orthogonal set $\{f_k\}$ is the orthogonal projection onto the space spanned by this set. This result is essential to understand the convergence of the Fourier series of f to f.

Chapter 6

Bessel's Inequality and Parseval's Theorem

To understand the structure of a Hilbert space, lets begin by looking at an example in the finite dimensional space, $\mathbb{R}^3$. It is often helpful to study a more concrete example and examine its characteristics and see if the same characteristics can be extended to the infinite case. First, consider a vector $< 4\widehat{i} - 3\widehat{j} + 2\widehat{k} >$ where $i =< 1, 0, 0 >$, $j =< 0, 1, 0 >$, and $k =< 0, 0, 1 >$ serve as an orthonormal basis. Now compute $< 4\widehat{i} - 3\widehat{j} + 2\widehat{k} > \cdot \widehat{j} = -3\widehat{j} \cdot \widehat{j} = -3 < \widehat{j} >^2 = -3$ Notice that only the coefficient of the $\widehat{j}$ survives. Interestingly, this result holds true for the inner product of an infinite series and an orthonormal sequence. The following theorem states this very result.

Theorem 6.1. *Suppose that $f = \sum_{k=1}^{\infty} c_k f_k$ for an orthonormal sequence $\{f_k\}_{k=1}^{\infty}$ in an inner product space $\mathcal{V}$. Then $c_k =< f, f_k >$ for each k.*

Which means that the coefficients of each k will equal the inner product of a vector and an orthonormal sequence. Furthermore, let $\{f_k\}_{k=1}^{\infty}$ be an orthonormal sequence in $\mathcal{V}$, and let $f \in \mathcal{V}$. Then the Fourier series of f with respect to $\{f_k\}_{k=1}^{\infty}$ is represented by $\sum_{k=1}^{\infty} < f, f_k > f_k$ and $< f, f_k >$ are the Fourier coefficients of f with respect to $\{f_k\}_{k=1}^{\infty}$. The proof of Theorem 6.1 can be found in [Sax01].

Now that we have identified these coefficients, we state a theorem which describes the size of these coefficients.

Theorem 6.2. *[Sax01] (Bessel's Inequality) Suppose that* $\{f_k\}_{k=1}^{\infty}$ *is an orthonormal sequence in an inner product space* $\mathcal{V}$*. For every* $f \in \mathcal{V}$*, the series (of nonnegative real numbers)* $\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq \|f\|^2$.

Proof. We will begin our proof by looking at the partial sum s_n of the Fourier series for f defined as $s_n = \sum_{j=1}^{n} \langle f, f_j \rangle f_j$. Then

$$\begin{aligned}
\langle f - s_n, f_k \rangle &= \langle f, f_k \rangle - \langle s_n, f_k \rangle \\
&= \langle f, f_k \rangle - \left\langle \sum_{j=1}^{n} \langle f, f_j \rangle f_j, f_k \right\rangle \\
&= \langle f, f_k \rangle - \sum_{j=1}^{n} \langle \langle f, f_j \rangle f_j, f_k \rangle \\
&= \langle f, f_k \rangle - \sum_{j=1}^{n} \langle f, f_j \rangle \langle f_j, f_k \rangle \\
&= \langle f, f_k \rangle - \sum_{j=1}^{n} \langle f, f_j \rangle \delta_{jk} \\
&= \langle f, f_k \rangle - \langle f, f_k \rangle = 0
\end{aligned}$$

Since the inner product is zero, we know that $f - s_n$ is orthogonal to each f_k. In addition

$$\begin{aligned}
\langle f - s_n, s_n \rangle &= \left\langle f - s_n, \sum_{k=1}^{n} \langle f, f_k \rangle f_k \right\rangle \\
&= \sum_{k=1}^{n} \langle f - s_n, \langle f, f_k \rangle f_k \rangle \\
&= \sum_{k=1}^{n} \overline{\langle f, f_k \rangle} \langle f - s_n, f_k \rangle = 0.
\end{aligned}$$

By the previous argument, we know $f - s_n$ is orthogonal to s_n. Thus,

$$\|f - s_n\|^2 + \|s_n\|^2 = \|f\|^2 .$$

Therefore,

$$\|s_n\|^2 \leq \|f\|^2 .$$

Recall

$$\|s_n\|^2 = \left\|\sum_{k=1}^{n} \langle f, f_k\rangle f_k\right\|^2 = \sum_{k=1}^{n} \|\langle f, f_k\rangle f_k\|^2 .$$

We know that

$$\|s_n\|^2 = \sum_{k=1}^{n} |\langle f, f_k\rangle|^2 \cdot \|f_k\|^2 = \sum_{k=1}^{n} |\langle f, f_k\rangle|^2 .$$

Hence

$$\sum_{k=1}^{n} |\langle f, f_k\rangle|^2 \leq \|f\|^2 .$$

And since this inequaltiy holds for every n,

$$\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq \|f\|^2 .$$

□

Bessel's Inequality states that the norm of the projection of f onto the span of $\{f_k\}$ is less than or equal to the norm of f itself. The following theorem helps us determine under what conditions Bessel's Inequality holds.

Theorem 6.3. *(Parseval's Theorem) Suppose that $\{f_k\}_{k=1}^{\infty}$ is an orthonormal sequence in an inner product space $\mathcal{V}$. Then $\{f_k\}_{k=1}^{\infty}$ is a complete orthonormal sequence if and only if for every $f \in \mathcal{V}$, $\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 = \|f\|^2$. $\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 = \|f\|^2$ is known as Parseval's Identity.*

Proof. Suppose $\{f_k\}_{k=1}^{\infty}$ is a complete orthonormal sequence in an inner product space $\mathcal{V}$ so that $f = \sum_{n=1}^{\infty} \langle f, f_k\rangle f_k$. Then we know $c_k = \langle f, f_k\rangle$ for each k and $\sum_{k=1}^{\infty} \left|\langle f, f_k\rangle^2\right| \leq \|f\|^2$. We need to show $\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \geq \|f\|^2$. Let $g = f - \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k = 0$ (by assumption). Then $\|g\| = \left\langle f - \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k, f - \sum_{j=1}^{\infty} \langle f, f_j\rangle f_j \right\rangle$
$=\langle f, f\rangle - \left\langle f, \sum_{j=1}^{\infty} \langle f, f_j\rangle f_j \right\rangle - \langle \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k, f\rangle + \left\langle \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \langle f, f_k\rangle f_k, \langle f, f_j\rangle f_j \right\rangle$
$=\langle f, f\rangle - \sum_{j=1}^{\infty} \overline{\langle f, f_j\rangle} \langle f, f_j\rangle - \sum_{k=1}^{\infty} \langle f, f_k\rangle \langle f_k, f\rangle + \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \langle f, f_k\rangle \overline{\langle f, f_j\rangle} \langle f_k, f_j\rangle$
$=\langle f, f\rangle - \sum_{j=1}^{\infty} \overline{\langle f, f_j\rangle} \langle f, f_j\rangle - \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 + \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \langle f, f_k\rangle \overline{\langle f, f_j\rangle} \delta_{kj}$.
Let $j = k$, then we have $\langle f, f\rangle - \sum_{k=1}^{\infty} \overline{\langle f, f_k\rangle} \langle f, f_k\rangle - \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 + \sum_{k=1}^{\infty} \langle f, f_k\rangle \overline{\langle f, f_k\rangle}$
$= \langle f, f\rangle - \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 - \|f\|^2 + \|f\|^2 = 0$ since f_n is a complete orthonormal sequence. Then, $\langle f, f\rangle - \sum_{k=1}^{\infty} \|f, f_k\|^2 = 0$. Thus, $\sum_{k=1}^{\infty} \langle f, f_k\rangle^2 = \|f\|^2$.

Now we will prove the reverse implication.

Given $\|f\|^2 = \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2$, we need to show that $\{f_k\}_{k=1}^{\infty}$ is a complete orthonormal

sequence. Given f, there exists a c_k such that $f = \sum_{k=1}^{\infty} c_k f_k = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k$.
Consider $g = f - \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k$ to show that $g = 0$.
Then $\|g\| = \left\langle f - \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, f - \sum_{j=1}^{\infty} \langle f, f_j \rangle f_j \right\rangle$
$= \langle f, f \rangle - \left\langle f, \sum_{j=1}^{\infty} \langle f, f_j \rangle f_j \right\rangle - \langle \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, f \rangle - \left\langle \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \langle f, f_k \rangle f_k, \langle f, f_j \rangle f_j \right\rangle$
$= \langle f, f \rangle - \sum_{j=1}^{\infty} \langle \overline{f, f_j} \rangle \langle f, f_j \rangle - \sum_{k=1}^{\infty} \langle f, f_k \rangle \langle f_k, f \rangle + \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \langle f, f_k \rangle \langle \overline{f, f_j} \rangle \delta_{kj}$.
Let $j = k$, then $\|g\| = \langle f, f \rangle - \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 - \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 - \sum_{k=1}^{\infty} \langle f, f_k \rangle \langle \overline{f, f_k} \rangle$
$= \|f\|^2 - \|f\|^2 - \|f\|^2 + \|f\|^2 = 0$. □

For L^2 functions, Parseval's Identity is comparable to the Pythagorean Theorem. Thus, Parseval's Identity is equivalent to the convergence of the Fourier sereis in the L^2-sense. We will continue discussing the convergence of the Fourier series in Chapter 7. The following examples illustrate the importance of these theorems.

Example 6.1. *Show that the classical Fourier series of $f(x) = x$ is:*

$$2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} sin(nx)$$

Lets determine the Fourier coefficients of f:

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) cos(kt) dt \text{ and } b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) sin(kt) dt$$

Thus,

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x cos(kx) dx = \frac{1}{\pi} \left[\frac{kx sin(kx) + cos(kx)}{k^2} \right]_{-\pi}^{\pi} = \frac{1}{\pi} \left[\frac{cos(k\pi) - cos(k\pi)}{k^2} \right] = 0$$

Therefore, $a_k = 0$ for all k. In addition,

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x sin(kx) dx = \frac{1}{\pi} \left[\frac{sin(kx) - kx cos(kx)}{k^2} \right]_{-\pi}^{\pi} = \frac{-2}{k} cos(k\pi)$$

$$b_k = \begin{cases} \frac{-2}{k} & : \text{k is even} \\ \frac{2}{k} & : \text{k is odd} \end{cases}$$

Thus, the classical Fourier series of $f(x) = x$ is given by

$$2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} sin(nx)$$

Example 6.2. *Use your work in example* 6.1*, together with Parseval's identity, to obtain Euler's remarkable identity*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Let $g(x) = sin(nx)$, then

$$\|g\|_2^2 = \int_{-\pi}^{\pi} sin(nx) \cdot sin(nx)dx = \int_{-\pi}^{\pi} (sin(nx))^2 dx = \int_{-\pi}^{\pi} \frac{1 - cos(2nx)}{2} dx$$

(Since $(sin(nx))^2 = \frac{1-cos(2nx)}{2}$)

$$= \int_{-\pi}^{\pi} \frac{1}{2}dx - \frac{1}{2}\int_{-\pi}^{\pi} cos(2nx)dx = \left[\frac{1}{2}x - \frac{1}{4n}sin(2nx)\right]_{-\pi}^{\pi} = \pi$$

Hence

$\|g\|_2 = \sqrt{\pi}$ and $f_n(x) = \frac{1}{\sqrt{\pi}} sin(nx)$ and $f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} sin(nx)$

$$\langle f, f_k \rangle = \left\langle 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} sin(nx), \frac{1}{\sqrt{\pi}} sin(kx) \right\rangle$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \langle sin(nx), sin(kx)\rangle$$

$(\langle sin(nx), sin(kx)\rangle = 0$ except when $n = k)$

Let $n = k$, then we have

$$\langle f, f_k \rangle = \frac{2}{\sqrt{\pi}} \frac{(-1)^{k+1}}{k} \langle sin(kx), sin(kx)\rangle = 2\sqrt{\pi}\frac{(-1)^{k+1}}{k}$$

Furthermore,

$$\|f\|_2^2 = \int_{-\pi}^{\pi} x^2 dx = \frac{1}{3}x^3|_{-\pi}^{\pi} = \frac{2\pi^3}{3}$$

And,

$$\sum_{k=1}^{\infty} \|\langle f, f_k\rangle\|^2 = 4\pi \sum_{k=1}^{\infty} \frac{\left|(-1)^{k+1}\right|}{k^2} = 4\pi \sum_{k=1}^{\infty} \frac{1}{k^2}$$

By Parseval's Identity,

$$4\pi \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{2\pi^3}{3}$$

so,

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Example 6.3. *Use your work in example* 6.1*, together with Parseval's identity, to obtain Euler's remarkable identity* $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

Let $g(x) = cos(nx)$, then

$$\|f\|_2^2 = \int_{-\pi}^{\pi} cos(nx) \cdot cos(nx)dx = \int_{-\pi}^{\pi} (cos(nx))^2 dx = \int_{-\pi}^{\pi} \frac{1 - sin(2nx)}{2} dx$$

(Since $(cos(nx))^2 = \frac{1-sin(2nx)}{2}$)

$$= \int_{-\pi}^{\pi} \frac{1}{2} dx - \frac{1}{2} \int_{-\pi}^{\pi} sin(2nx)dx = \left[\frac{1}{2}x + \frac{1}{4n} cos(2nx)\right]_{-\pi}^{\pi} = \pi$$

Hence $\|g\|_2 = \sqrt{\pi}$ and $f_n(x) = \frac{1}{\sqrt{\pi}} cos(nx)$ and $f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} cos(nx)$ and

$$\langle f, f_k \rangle = \left\langle \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} cos(nx), \frac{1}{\sqrt{\pi}} cos(kx) \right\rangle$$

Considering the coefficient components of $f(x) = x^2$ with respect to the orthonomal basis we have $x_0 = \frac{\sqrt{2\pi}\pi^2}{3}$ and $x_n = \frac{4(-1)^n\sqrt{\pi}}{n^2}$ for $n = 1, 2, ...$
Let $n = k$, we have

$$x^2 = \frac{\sqrt{2\pi}\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n\sqrt{\pi}}{n^2} (\frac{1}{\sqrt{\pi}} cos(nx))$$

Applying Parseval's Identity: $\int_{-\pi}^{\pi} x^4 dx = \sum_{n=0}^{\infty} (x_n)^2$, we get

$$\frac{2}{5}\pi^5 = (\frac{\sqrt{2\pi}\pi^2}{3})^2 + \sum_{n=1}^{\infty} \frac{16\pi}{n^4} = \frac{2\pi^5}{9} + \sum_{n=1}^{\infty} \frac{16\pi}{n^4}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Chapter 7

Convergence of the Fourier Series in the L^2-norm

There are certain criteria that must be met to insure that the Fourier series of f converges to f. After looking at several functions and studying the convergence properties of their Fourier series, we see that the Fourier series of f, for f square integrable, converges to f in mean (or in the L^2-norm). We show this result in this chapter.

The outline of this chapter is as follows: We begin with an orthonormal sequence, $\{f_k\}_{k=1}^{\infty}$, in an inner product space $\mathcal{V}$, and defined an element $f = \sum_{k=1}^{\infty} c_k f_k$ where $c_k = \langle f, f_k \rangle$. We call $\sum_{k=1}^{\infty} \langle f, f_k \rangle f_k$ the Fourier series of f with respect to $\{f_k\}_{k=1}^{\infty}$ and $\langle f, f_k \rangle$ the Fourier coefficients of f with respect to $\{f_k\}_{k=1}^{\infty}$. Next, we obtain Bessel's Inequality which tells us when f is square integrable, the series (on nonnegative real numbers) $\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2$ converges and $\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq \|f\|^2$. Which is telling us that the norm of the projection of f onto the span of $\{f_k\}$ is less than or equal to the norm of f itself. Now assuming we have a complete orthonormal sequence, we can replace the inequality with an equality and obtain Parseval's Identity. In the special case where $f \in L^2([-\pi, \pi])$, the sum of squares of the Fourier coefficients of f with respect to the trigonometric system $\{f_k\}_{k=1}^{\infty}$, is always finite. This result is stated in the following theorem.

Theorem 7.1. *[Sax01] Assume that*

1. $\{d_k\}_{k=1}^{\infty}$ *is a sequence of real numbers such that* $\sum_{k=1}^{\infty} d_k^2$ *converges, and*

2. $\mathcal{V}$ *is a Hilbert space with complete orthonormal sequence* $\{f_k\}_{k=1}^{\infty}$.

Then there is an element $f \in \mathcal{V}$ *whose Fourier coefficients with respect to* $\{f_k\}_{k=1}^{\infty}$ *are the numbers* d_k *and*

$$\|f\|^2 = \sum_{k=1}^{\infty} d_k^2$$

Proof. Define $s_n = \sum_{k=1}^{n} d_k f_k$. For $m > n$, the square of the distance between s_n and s_m is as follows:

$$\|s_n - s_m\|^2 = \sum_{j=n+1}^{m} \sum_{k=n+1}^{m} d_j d_k \langle f_j, f_k \rangle = \sum_{k=n+1}^{m} d_k^2.$$

This is true since when $j = k$, $\langle f_j, f_k \rangle = 1$. Hence, $s_n = \sum_{k=1}^{n} d_k f_k$ is Cauchy. By assumption, if $f \in \mathcal{V}$, which is a Hilbert space, then there is an $f \in \mathcal{V}$ such that

$$\lim_{n \to \infty} \|s_n - f\| = 0.$$

Therefore, $f = \sum_{k=1}^{\infty} d_k f_k$ and $d_k = \langle f, f_k \rangle$. Since Parseval's theorem states $\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 = \|f\|^2$, it follows that $\|f\|^2 = \sum_{k=1}^{\infty} d_k^2$.

□

This Theorem tells us that the sum of squares of the Fourier coefficients ($d_k = \langle f, f_k \rangle$) is finite. Furthermore, Parseval's Identity tells us that if we have a complete orthonormal sequence (which we have in a Hilbert space), then the inequality in Bessel's Inequality is replaced with an equality. Hence, by Parseval's Identity we know $\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 = \|f\|^2$. We will see that Parseval's Identity has important consequences for the completeness of the Fourier series whose orthogonal basis is the Trigonometric system.

We want to show that the Fourier series of f converges in mean to f. In order to do this, we need the aid of the following theorem.

Theorem 7.2. *[Sax01] For an orthonormal sequence $\{f_k\}_{k=1}^{\infty}$ in $L^2([-\pi,\pi],m)$, the following are equivalent:*

1. *$\{f_k\}_{k=1}^{\infty}$ is a complete orthonormal sequence.*
2. *For every $f \in L^2$ and $\epsilon > 0$ there is a finite linear combination*
$$g = \sum_{k=1}^{n} d_k f_k$$
such that $\|f - g\| \leq \epsilon$
3. *If the Fourier coefficients with respect to $\{f_k\}_{k=1}^{\infty}$ of a function in L^2 are all 0, then the function is equal to 0 almost everywhere.*

Proof. It follows from the definition that (1) implies (2). We will prove that (2) implies (3). Let f be a square integrable function such that $\langle f, f_k \rangle = 0$ for all k. Let $\epsilon > 0$ be given and choose g as in (2). Then

$$\|f\|_2^2 = \left| \|f\|_2^2 - \left\langle f, \sum_{k=1}^{n} d_k f_k \right\rangle \right| = |\langle f, f \rangle, \langle f - g \rangle| = |\langle f, f - g \rangle|$$

(Where $g = \sum_{k=1}^{n} d_k f_k$). Hence

$$\|f\|_2^2 \leq \|f\|_2 \cdot \|f - g\|_2 \leq \epsilon \|f\|_2$$

(By the Schwartz Inequality)
Thus, $\|f\|_2 \leq \epsilon$. Since ϵ was arbitrary, we assert that f must be 0 almost everywhere.
To prove (3) implies (1), let $f \in L^2$ and put

$$s_n = \sum_{k=1}^{n} \langle f, f_k \rangle f_k.$$

Since $\{s_n\}_{n=1}^{\infty}$ is Cauchy in L^2 (by Theorem 6.2), there is a function $g \in L^2$ such that

$$\lim_{n \to \infty} \|s_n - g\|_2 = 0$$

We have,

$$g = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k.$$

Since the Fourier coefficients of g are the same as the Fourier coefficients of f with respect to $\{f_k\}_{k=1}^{\infty}$, that is, $\langle g, f_k\rangle = \langle f, f_k\rangle$, we have that $f-g$ has zero Fourier coefficients with respect to f_k, thus, by (2), $f-g$ must equal 0 almost everywhere. Hence,

$$f = \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k.$$

Since f was arbitrary, $\{f_k\}_{k=1}^{\infty}$ is a complete orthonormal sequence.

□

We are ready to prove the Riesz-Fischer theorem, which asserts the completeness of the trigonometric system.

Theorem 7.3. *Riesz-Fischer Theorem: [Sax01] and [Kat76] The trigonometric system*

$$\frac{1}{\sqrt{2\pi}}, \frac{cos(nx)}{\sqrt{\pi}}, \frac{sin(mx)}{\sqrt{\pi}}, n, m = 1, 2, ...,$$

forms a complete orthonormal sequence in $L^2([-\pi,\pi], m)$. That is, if f is such that $|f|^2$ is Lebesgue integrable, then its (classical) Fourier series converges to f. The convergence is convergence in the norm $\|\cdot\|_2$, i.e.

$$\lim_{n\to\infty}\left[\int_{-\pi}^{\pi}\left[f(x) - \left(\frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos(kx) + \sum_{k=1}^{n} b_k\sin(kx)\right)\right]^2 dx\right] = 0$$

Proof. We have previously established that the trigonometric system is orthonormal. We need to show that the trigonometric system is complete. We will use Theorem 7.2 part (3) to complete the proof.

Let f be a continuous real-valued function where $\langle f, f_k\rangle = 0$ for each f_k. If $f \neq 0$, then there exists an x_0 at which $|f|$ achieves a maximum. We also know that $f(x_0) > 0$. Let δ be small enough such that $f(x) > \frac{f(x_0)}{2}$ for all x in the interval $(x_0 - \delta, x_0 + \delta)$. Consider the following function

$$t(x) = 1 + cos(x_0 - x) - cos(\delta)$$

This function is an example of a trigonometric polynomial where $t(x)$ is a finite linear combination of the functions in the trigonometric system. The following are true:

1. $1 < t(x)$, for all x in $(x_0 - \delta, x_0 + \delta)$

2. $|t(x)| \leq 1$ for all x outside of $(x_0 - \delta, x_0 + \delta)$.

We have shown that f is orthogonal to every member of the trigonometric system and thus f is orthogonal to every trigonometric polynomial. Particularly, f is orthongonal to t^n for every positive integer n. We will prove that $f = 0$ leads to a contradiction. We have

$$0 = \langle f, t^n \rangle = \int_{-\pi}^{\pi} f(x)t^n(x)dx$$

$$= \int_{-\pi}^{x_0-\delta} f(x)t^n(x)dx + \int_{x_0-\delta}^{x_0+\delta} f(x)t^n(x)dx + \int_{x_0-\delta}^{\pi} f(x)t^n(x)dx.$$

The first and third integrals are bounded in absolute value for each n by $2\pi f(x_0)$ by (2) above. However, the middle integral is greater than or equal to $\int_a^b f(x)t^n(x)dx$, where $[a, b]$ is any closed interval in $(x_0 - \delta, x_0 + \delta)$. We know that t achieves a minimum value, m, since t is continuous on $[a, b]$. Since (1) tells us that $m > 1$, we have

$$\int_a^b f(x)t^n(x)dx \geq \frac{f(x_0)}{2} \cdot m^n \cdot (b - a),$$

which grows without bound as $n \to \infty$. This is a contradiction of the assumption that $0 = \langle f, t^n \rangle$ for all n. Hence, any continuous real-valued function that is orthogonal to every trigonometric polynomial must be identically zero.

Next we will look at the case where f is continuous but not real-valued. According to our hypothesis, we have

$$\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = 0, k = 0, \pm 1, \pm 2, ...,$$

and also

$$\int_{-\pi}^{\pi} \overline{f(x)}e^{-ikx}dx = 0, k = 0, \pm 1, \pm 2, ...$$

Since the real and imaginary parts of f are orthogonal to each of the members of the trigonometric system, the real and imaginary parts of f are identically zero. Thus, f is identically zero.

Now we consider when f is not continuous. Define the continuous function

$$F(x) = \int_{-\pi}^{x} f(t)dt.$$

Let $f_k(x) = \frac{cos(kx)}{\sqrt{\pi}}$. Applying our hypothesis, we have

$$0 = \int_{-\pi}^{\pi} f(x)cos(kx)dx.$$

After integrating by parts, we have,

$$\int_{-\pi}^{\pi} F(x)sin(kx)dx = \frac{1}{k}\int_{-\pi}^{\pi} f(x)cos(kx)dx = 0$$

Likewise, we can show that

$$\int_{-\pi}^{\pi} F(x)cos(kx)dx = 0.$$

Since F and hence, $F - C$, for every constant C, is orthogonal to each of the nonconstant members of the trigonometric system. Lastly, we take a look at the term $\frac{1}{\sqrt{2\pi}}$. Take

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)dx.$$

Then $F - C_0$ is orthogonal to every member of the trigonometric system. Since F is continuous, then $F - C_0$ is contiuous, and we have that $F - C_0$ is identically zero. Therefore, we have that $f = F'$ is zero almost everywhere.

$\square$

This type of convergence is often called "in mean" convergence and does not imply either uniform or pointwise convergence. There is no uniform convergence in general. This is because the partial sums of the Fourier series of f are always continuous functions, and if the convergence of the series were uniform, then f would have to be continuous which is not always the case. Therefore, it is not always possible to achieve uniform convergence because $L^2([-\pi,\pi], m)$ contains discontinuous functions.

Chapter 8

Conclusion

In this thesis we studied the structure of a Hilbert space in order to see that the Fourier series of a square summable function converges in the mean to its corresponding function. Hilbert spaces, such as $L^2([-\pi, \pi])$, play an important role in applications including Fourier analysis, harmonic analysis, and quantum physics. The geometric intuitions that accompany Euclidean spaces are sucessfuly generalized to corresponding situations in Hilbert spaces.

In Chapter 2, we included important definitions and terminology that were necessary to gain understanding of Hilbert spaces, Fourier series, and the types of convergence discussed in this thesis.

In Chapter 3, we included some examples of Hilbert spaces for the purpose of demonstrating their structure in order to understand the proof of the convergence of Fourier series of square summable functions that follows in Chapter 7.

Chapter 4 investigated the question of when does a Fourier series converge to its function and if it does converge, what type of convergence. Uniform, pointwise, and mean convergence were the three types of convergence discussed in this chapter. After defining and proving each type of convergence, we were able to include examples that illustrated the relationship between these types of convergence. Since the Fourier series was crucial to our topic of study, we computed the classical Fourier series for an even and an odd function. We found that uniform and pointwise convergence failed in both of these cases. This result motivated us to further investigate mean convergence. Lastly, we illustrated how to apply the Gram-Schmidt process to obtain an orthonormal system.

This process was applied to the fucntions $1, x, x^2, x^3$ in order to obtain the first four Legendre polynomials.

Chapter 5 featured the Projection Theorem and its Lemma which are paramount to understanding the completeness of the Hilbert space and the convergence of the Fourier series of f to f. We were able to prove uniqueness in both the Lemma and the Projection Theorem.

In Chapter 6, we begin with defining the Fourier coefficients of f with respect to the orthonormal sequence $\{f_k\}_{k=1}^{\infty}$. Next we investigated the size of these Fourier coefficients by studying Bessel's Inequality. Furthermore, when we have a complete orthonormal space (i.e. a Hilbert space), Parseval's Theorem turns Bessel's Inequality into an equality. The proof of Parseval's Theorem is included in this chapter. This result leads to the conclusion that the convergence of the Fourier series in L^2 is equivalent to Parseval's Identity. Also included in this chapter are examples that illustrate how to use the Fourier series of f and Parseval's Identity to obtain two of Euler's identities.

In Chapter 7, we included the necessary theorems that lay the ground work for the very important Riesz-Fischer Theorem. Hence, we were able to show that the Trigonometric system forms a complete orthonormal system in $L^2([-\pi, \pi])$ and thus its Fourier series converges to f in the norm.

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